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Separable coordinate systems for the Hamilton–Jacobi, Klein–Gordon and wave equations in curved spaces

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Abstract. There are exactly two types of separable coordinates for the Hamilton–Jacobi, Klein–Gordon and wave equations. One type can be reduced to separable coordinates adapted to a (conformal) Killing vector, the other type to orthogonal coordinates adapted to eigenvectors of a (conformal) Killing tensor. We derive a canonical form of the metric tensor which is a necessary and sufficient condition for the existence of a separable coordinate system for the Hamilton–Jacobi equation. For the Klein–Gordon equation the metric is further restricted by a condition on the Ricci tensor. We give also sufficient conditions for the existence of separable coordinates in terms of linear or quadratic constants of motion.

1. Introduction

Many authors have studied coordinate systems in which the Hamilton–Jacobi equation separates completely. This problem was first investigated by Liouville and later by Stäckel (1890, 1891, 1893), Levi-Cività (1904), Dall'Aqua (1908, 1912) and Burgatti (1911). They gave metrics in which the Hamilton–Jacobi equation separates completely and found connexions between separable systems for the Hamilton–Jacobi equation and the existence of constants of motion. Stäckel was the first who pointed out that quadratic constants of motion are related to orthogonal separable systems. Robertson (1927) and Eisenhart (1934) discussed the complete separability of the Klein–Gordon and wave equations in spaces admitting a complete set of mutually orthogonal families of hypersurfaces. They found the same results as Stäckel for the Hamilton–Jacobi equation and additional conditions for the Ricci tensor. Havas (1975) gave the most general metric admitting coordinate systems in which the Hamilton–Jacobi and Schrödinger equations separate completely.

A newer motivation for relativists to investigate separation problems is the work done by Carter (1968) concerning the separability in some type {22} space–times, and especially the beautiful paper of Woodhouse (1975). He shows the relationship between Killing tensors and separable systems for the Hamilton–Jacobi equation: there are essentially two types of separable systems for the (massless) Hamilton–Jacobi equation, whose separable coordinate is adapted either to a (conformal) Killing vector or to an eigenvector of a (conformal) Killing tensor.

In this paper we extend Woodhouse's results to the Klein–Gordon (wave) equation and show that there are again only these two types of separable systems, such that the separable coordinate is again adapted either to a Killing vector (generator of a homothety) or to an eigenvector of a (conformal) Killing tensor.

We find canonical metric tensors and additional conditions for the corresponding Ricci tensor in such a way that at least one separable coordinate exists.

Finally, we derive sufficient conditions for the existence of a separable system for the Klein–Gordon or wave equations in terms of constants of motion.

2. Definitions and conventions

(M, g) is a smooth n dimensional Riemannian or pseudo-Riemannian manifold with a positive definite or Lorentz metric. A set of local coordinates is denoted by (x^a). Except where otherwise indicated, Latin indices a, b, c run from 1 to n, Greek indices α, β, γ from 2 to n. The determinant of the covariant metric tensor is denoted by g. The *Hamilton–Jacobi equation* (HJ) is given by

$$g^{ab}\,\partial_a S\,\partial_b S - m^2 = 0 \tag{1}$$

the *Klein–Gordon equation* (KG) by

$$\Psi^{-1} g^{ab}\,\partial_a\,\partial_b \Psi + \Psi^{-1}\frac{\partial_b(\sqrt{|g|}\,g^{ab})}{\sqrt{|g|}}\partial_a \Psi - m^2 = 0 \tag{2}$$

which is called the *wave equation* (WE) if and only if $m = 0$.

In the following we investigate the conditions for the existence of a coordinate system (x^a) in which the HJ or KG and WE separate with respect to x^1, that is to say the *ansatz*

$$S(x^a) = S_1(x^1) + S_2(x^\alpha) \qquad \text{for } S \tag{3}$$

or

$$\Psi(x^a) = \Phi(x^1)\chi(x^\alpha) \qquad \text{for } \Psi \tag{4}$$

inserted into (1) or (2) causes those equations, after multiplication with a suitable function $U = U(x^a)$, to split up into a partial differential equation in x^α for $S_2(x^\alpha)$ or $\chi(x^\alpha)$, and an ordinary differential equation in x^1 for $S_1(x^1)$ or $\Phi(x^1)$. A necessary condition for the separability of the HJ, KG and WE with respect to x^1 is that the coefficient Ug^{11} of $\partial_1 S_1\,\partial_1 S_1$ or $\partial_1\,\partial_1\chi$ is a function of x^1 alone, say $f(x^1)$. The transformation $x^1 \to \bar{x}^1(x^1) = \int^{x^1} f^{-1/2}(y)\,\mathrm{d}y$ gives, after dropping the bar, $(g^{\prime\prime} \neq 0)$

$$U = (g^{11})^{-1}$$

and we obtain for (1) and (2) multiplied by U

$$\frac{g^{ab}}{g^{11}}\partial_a S\,\partial_b S - \frac{m^2}{g^{11}} = 0 \tag{5}$$

and

$$\Psi^{-1}\frac{g^{ab}}{g^{11}}\partial_a\,\partial_b\Psi + \Psi^{-1}\frac{\partial_a(\sqrt{|g|}\,g^{ab})}{\sqrt{|g|}\,g^{11}}\,\partial_b\Psi - \frac{m^2}{g^{11}} = 0. \tag{6}$$

In the following, we assume that the transformation $x^1 \to \bar{x}^1(x^1)$, previously mentioned, is performed.

(x^a) is a *separable system* for the HJ, KG and WE if the corresponding equation separates with respect to x^1. We call x^1 the *separable coordinate.* All considerations are strictly local in that the set of local solutions of the differential equations is not restricted by identifications or boundary conditions. This enables us to consider S_1, S_2 or Φ, χ and their derivations as independent functions.

Some definitions concerning Killing tensors and constants of motions are needed (see Geroch 1970, Sommers 1973, Woodhouse 1975).

The Nijenhuis bracket operation is defined by

$$[A,B]^{a_1\cdots a_{p+q+1}} := pA^{b(a_1\cdots a_{p-1}}\nabla_b B^{a_p\cdots a_{p+q+1})} - qB^{b(a_1\cdots a_{q-1}}\nabla_b A^{a_q\cdots a_{p+q+1})} \tag{7}$$

where $A^{a\cdots b}$ and $B^{a\cdots c}$ are symmetric contravariant tensor fields of order p and q respectively and ∇ is any torsion-free connexion on (M, g).

We attach to each symmetric contravariant tensor field $A^{a\cdots b}$ a function $A(x^a, p_a) := A^{a\cdots b}p_a \ldots p_b$ on the covariant tangent bundle T^*M where p^a is the tangent on the curve $x^a = x^a(s)$. The Poisson bracket of two functions on T^*M is

$$\{A, B\} := \frac{\partial A}{\partial p_a}\frac{\partial B}{\partial x^a} - \frac{\partial B}{\partial p_a}\frac{\partial A}{\partial x^a}.$$

The Nijenhuis bracket and the Poisson bracket are closely related.

Proposition 1. Let $A^{a\cdots b}$ and $B^{a\cdots c}$ be symmetric contravariant tensor fields and $C^{a\cdots d} = [A, B]^{a\cdots d}$. Then the associated functions fulfil $C = \{A, B\}$.

This is easy to prove by direct computation from (7). The symmetric tensor $C^{a\cdots c}$ is a *conformal Killing tensor* if a tensor field $B^{a\cdots b}$ exists such that

$$[C, g]^{a\cdots d} = B^{(a\cdots b}g^{cd)}.$$

With proposition 1, we obtain $\{C, H\} \sim H$ where $H = \frac{1}{2}g^{ab}p_a p_b$ denotes the Hamiltonian. If a symmetric tensor $K^{a\cdots b}$ satisfies

$$[K, g]^{a\cdots d} = 0$$

it is called a *Killing tensor.* Then $\{K, H\} = 0$.

Corollary 1. The metrics g_{ab} and $\hat{g}_{ab}$ are conformal: $\hat{g}_{ab} = Wg_{ab}$. K is a constant of motion with respect to $\hat{g}_{ab}$. If H is the Hamiltonian of g_{ab} then $\{K, H\} \sim H$.

Proof. The Hamiltonians H of g_{ab} and $\hat{H}$ of $\hat{g}_{ab}$ are related by $H = W\hat{H}$. Therefore $\{K, H\} = \{K, W\}\hat{H} + W\{K, \hat{H}\} = \{K, W\}\hat{H} = W^{-1}\{K, W\}H \sim H$.

ξ^a is called a *conformal Killing vector* if and only if

$$[\xi, g]^{ab} = fg^{ab} \tag{8}$$

with $f = n^{-1}\nabla_a\xi^a$. If f = constant, ξ^a generates a *homothety*. If $f = 0$, ξ^a defines a *Killing vector.* If we choose the parameter t on the integral curves (orbits) generated by ξ^a as coordinate, t is adapted to the (conformal) Killing vector and we obtain

$$\frac{\partial}{\partial t}g^{ab} = 0 \tag{9a}$$

if ξ^a is a Killing vector, and

$$\frac{\partial}{\partial t}(Vg^{ab})=0 \qquad \text{with} \qquad f=\frac{\partial}{\partial t}\ln|V| \tag{9b}$$

if ξ^a is a conformal Killing vector. A *quadratic* (*conformal*) *Killing tensor* is a (conformal) Killing tensor of order two. We will now generalize a theorem due to Carter (1973) concerning quadratic constants of motion.

Proposition 2. $(x^a)=(x^i, x^j, x^r)$ are local coordinates with $i=1,\ldots,k; j=k+1,\ldots,l; r=k+l+\lambda,\ldots,n$ and the Hamiltonian H takes the form

$$H=\frac{1}{2}\frac{H_1+H_2}{U_1+U_2}$$

with $U_1=U_1(x^i)$; $U_2=U_2(x^j)$ and $H_1=H_1(x^i, p_i, p_r)$; $H_2=H_2(x^j, p_j, p_r)$. Then

$$K=\frac{1}{2}\frac{U_1H_2-U_2H_1}{U_1+U_2}$$

is a quadratic constant of motion.

Proof. With $U:=U_1+U_2$ we find

$$4\{H, K\}=\frac{H_2}{U^2}\{H_1, U_1\}-\frac{H_1}{U^2}\{H_2, U_2\}+U\left\{\frac{H_1}{U}, \frac{H_2}{U}\right\}.$$

We compute $\{H_1, U^{-1}\}$ and $\{H_1, U_1\}$ explicitly, compare the results and find

$$\{H_1, U^{-1}\}=-U^{-2}\{H_1, U_1\}. \tag{10}$$

The same relation is valid for H_2 and U_2. From

$$\left\{\frac{H_1}{U}, \frac{H_2}{U}\right\}=\frac{H_2}{U}\{H_1, U^{-1}\}-\frac{H_1}{U}\{H_2, U^{-1}\}$$

we find together with (10)

$$\left\{\frac{H_1}{U}, \frac{H_2}{U}\right\}=-\frac{H_2}{U^2}\{H_1, U_1\}+\frac{H_1}{U^2}\{H_2, U_2\}$$

which inserted into the first equation yields the desired result.

The vector v^a is an *eigenvector* and the 1-form v_a *eigenform* of a symmetric tensor T^{ab} if

$$T_{ab}v^b=\lambda g_{ab}v^b \qquad \text{and} \qquad T^{ab}v_b=\lambda g^{ab}v_b.$$

It is worth noticing that in pseudo-Riemannian spaces a symmetric tensor need not be diagonalized.

x^1 is said to be a *trivial separable* coordinate for the HJ with $m\neq 0$ and KG (massless HJ; WE) if and only if it is adapted to a Killing vector (conformal Killing vector; generator of a homothety). (x^a) is then a trivial separable system. x^1 is called an *orthogonal coordinate* (with respect to the $n-1$ coordinates (x^α)) if $g^{1\alpha}=0$ (see Eisenhart 1948). x^1 is an *orthogonal separable* coordinate if it is orthogonal and separable. (x^a) is then an orthogonal separable system.

3. Basic types of separable systems

In this section we derive the two basic types of separable systems for the HJ, KG and WE. We assume that none of the $n-1$ coordinates (x^α) is adapted to a (conformal) Killing vector. We divide the separation *ansatz* (3) for the HJ and (4) for the KG and WE (6) into two classes:

(*a*)

$$S(x^a)=kx^1+S_2(x^\alpha) \qquad \text{with a real constant } k \tag{11}$$

$$\Psi(x^a)=\exp(cx^1)\chi(x^\alpha) \qquad \text{with a complex constant } c \tag{12}$$

(*b*)

$$S(x^a)=S_1(x^1)+S_2(x^\alpha) \qquad \text{with } S_1(x^1)\neq kx^1 \tag{13}$$

$$\Psi(x^a)=\Phi(x^1)\chi(x^\alpha) \qquad \text{with } \Phi(x^1)\neq \exp(cx^1). \tag{14}$$

k and c are arbitrary constants because $S(x^a)$ and $\Psi(x^a)$ represent complete solutions separating with respect to x^1.

Case (*a*). We insert *ansatz* (11) into the HJ (5) and obtain

$$k^2+2k\frac{g^{1\alpha}}{g^{11}}\partial_\alpha S_2+\frac{g^{\alpha\beta}}{g^{11}}\partial_\alpha S_2\,\partial_\beta S_2-\frac{m^2}{g^{11}}=0$$

from which we derive the separation condition $\partial_1 g^{ab}=0$ (and $\partial_1 g^{ab}/g^{11}=0$ for $m=0$) remembering the definition of separation and using the fact that k is an arbitrary constant. So x^1 is trivial separable for the HJ. Conversely, if x^1 is adapted to a (conformal) Killing vector, *ansatz* (11) causes the (massless) HJ to separate with respect to x^1.

With *ansatz* (12) we find for the KG

$$\frac{1}{\chi}\left(\frac{g^{\alpha\beta}}{g^{11}}\partial_\alpha\,\partial_\beta+\frac{\partial_b(\sqrt{|g|}g^{b\alpha})}{\sqrt{|g|}g^{11}}\partial_\alpha+2c\frac{g^{1\alpha}}{g^{11}}\partial_\alpha\right)\chi+c\left(\partial_1\ln|g^{11}\sqrt{|g|}|+\frac{\partial_\alpha(\sqrt{|g|}g^{1\alpha})}{\sqrt{|g|}g^{11}}\right)-\frac{m^2}{g^{11}}+c^2=0$$

and obtain the necessary and sufficient condition $\partial_1 g^{ab}=0$ for separability of the KG with respect to x^1 because c is an arbitrary constant and χ and its derivatives are independent functions. The separability conditions for the WE are

$$\partial_1\frac{g^{ab}}{g^{11}}=0 \qquad \text{and} \qquad \partial_b\,\partial_1\ln|g^{11}|=0$$

which means that $\partial/\partial x^1$ generates a homothety. We collect the results in the following proposition.

Proposition 3. We choose *ansatz* (11) for the HJ and (12) for the KG and WE. x^1 is a separable coordinate for the HJ, KG and WE if and only if x^1 is a trivial separable coordinate.

We may easily extend this last proposition to

Theorem 1. There exist q $(0\leqslant q\leqslant n)$ trivial separable coordinates (x^u) with

$u = 1, \ldots, q$ for the massive HJ and KG (massless HJ; WE) if and only if the coordinates (x^u) are adapted to q commuting Killing vectors (conformal Killing vectors; generators of homotheties).

The unique successful *ansatz* for the HJ is then

$$S(x^a) = k_u x^u + S_2(x^v)$$

for the KG and WE

$$\Psi(x^a) = \exp(c_u x^u)\chi(x^v)$$

where k_u are arbitrary real and c_u arbitrary complex constants and v runs from $q+1$ to n. The most general transformation $x^a = x^a(\bar{x}^b)$ such that at least $\bar{x}^1$ is again a separable coordinate for the transformed equations, is

$$x^u = A^u(\bar{x}^1) + B^u(\bar{x}^\alpha)$$
$$x^v = B^v(\bar{x}^\alpha) \tag{15}$$

where A^u and B^a are arbitrary non-vanishing functions. All separable systems related to trivial separable systems by transformation (15) are called, following Woodhouse (1975), *type II separable.*

Case (b). *Ansatz* (13) or (14) inserted into the HJ (5) or KG and WE (6) yields the term $g^{1\alpha}\,\partial_1 S_1\,\partial_\alpha S_2$ in the HJ or $g^{1\alpha}\,\partial_1 \ln|\Phi|\partial_\alpha \ln|\chi|$ in the KG and WE where S_1 and Φ, S_2 and χ are functions of x^1 and (x^α), respectively. Therefore $g^{1\alpha} = 0$ is a necessary condition for separability with respect to x^1 and we obtain for the HJ (5).

$$\partial_1 S_1\,\partial_1 S_1 + \frac{g^{\alpha\beta}}{g^{11}}\,\partial_\alpha S_2 \partial_\beta S_2 - \frac{m^2}{g^{11}} = 0 \tag{16}$$

and for the KG (6)

$$\frac{1}{\Phi}[\partial_1\,\partial_1 + \partial_1(\ln|g^{11}\sqrt{|g|}|)\,\partial_1]\Phi + \frac{1}{\chi}\left(\frac{g^{\alpha\beta}}{g^{11}}\,\partial_\alpha\,\partial_\beta + \frac{\partial_\beta(g^{\alpha\beta}\sqrt{|g|})}{g^{11}\sqrt{|g|}}\,\partial_\alpha\right)\chi - \frac{m^2}{g^{11}} = 0. \tag{17}$$

The massive HJ (16) separates with respect to x^1 if the second term depends only on (x^α) and the third term splits up into a sum of a function of x^1 and a second function of (x^α) equivalent with

$$g^{1\alpha} = 0 \tag{18a}$$

$$\partial_1 \frac{g^{\alpha\beta}}{g^{11}} = 0 \tag{18b}$$

$$\partial_\alpha\,\partial_1 (g^{11})^{-1} = 0. \tag{18c}$$

These conditions prove also to be sufficient. For the massless HJ they reduce to (18*a*, *b*) because $m = 0$. In the case of the KG (WE) we have to investigate two further terms. (17) separates with respect to x^1 if the first bracket depends on x^1, the second on (x^α) and the last term splits up into a sum of a function of x^1 and a second function of (x^α). Because Φ, χ and their derivatives are independent functions, we obtain the separability conditions (18), $\partial_1\,\partial_\alpha(\ln|g^{11}\sqrt{|g|}|) = 0$ and $\partial_1[(\sqrt{|g|}g^{11})^{-1}\,\partial_\beta(\sqrt{|g|}g^{\alpha\beta})] = 0$ where the last condition depends on the others. Therefore (18) and

$$\partial_1\,\partial_\alpha(\ln|g^{11}\sqrt{|g|}|) = 0 \tag{19}$$

are necessary so that the KG separates with respect to x^1. (18) and (19) are also sufficient. The conditions reduce to (18*a*, *b*) and (19) for the WE because $m=0$. Let $W^{\alpha\beta}(x^\gamma) := g^{\alpha\beta}(g^{11})^{-1}$ and $W_{\alpha\beta}=(W^{\alpha\beta})^{-1}$ and integrate (18*c*) to

$$1/g^{11}=U_1(x^1)+U_2(x^\alpha) \tag{20}$$

with arbitrary functions U_1 and U_2. Then (18) is equivalent with the form

$$ds^2=(U_1(x^1)+U_2(x^\alpha))[(dx^1)^2+W_{\alpha\beta}(x^\gamma)\,dx^\alpha\,dx^\beta] \tag{21}$$

of the metric. In the case of the massless HJ, (18*c*) drops and we obtain

$$ds^2=U(x^a)[(dx^1)^2+W_{\alpha\beta}(x^\gamma)\,dx^\alpha\,dx^\beta]$$

with arbitrary $U=U(x^a)$. These two metrics are conformal to

$$d\sigma^2=\gamma_{ab}\,dx^a\,dx^b=(dx^1)^2+W_{\alpha\beta}(x^\gamma)\,dx^\alpha\,dx^\beta. \tag{22}$$

If the conformal factor is written V, (19) is equivalent with

$$R_{1\alpha}=\frac{n-2}{4V}\partial_1\,\partial_\alpha V.$$

(The proof for that equivalence is identical with that of theorem 3 if $p=0$.) If (18*c*) is valid and equivalent with (20), then $V=U_1(x^1)+U_2(x^\alpha)$ and therefore $R_{1\alpha}=0$. That proves

Proposition 4. None of the $n-1$ coordinates (x^α) is adapted to a (conformal) Killing vector. x^1 is orthogonal separable

(i) for the massive HJ if and only if the metric is conformal to (22) with the conformal factor (20);
(ii) for the massless HJ if and only if the metric is conformal to (22);
(iii) for the KG if and only if (i) is valid and $R_{1\alpha}=0$;
(iv) for the WE if and only if (ii) is valid and $R_{1\alpha}=(n-2)(4V)^{-1}\,\partial_1\,\partial_\alpha V$ where V is the conformal factor and R_{ab} the Ricci tensor of (22).

The most general transformation $x^a=x^a(\bar{x}^b)$ such that $\bar{x}^1$ is separable again for the transformed equations is

$$x^1=A^1(\bar{x}^1) \qquad \text{and} \qquad x^\alpha=B^\alpha(\bar{x}^\beta) \tag{23}$$

with arbitrary non-vanishing functions A^1 and B^α.

The connexion between the orthogonal separable systems derived and constants of motion is given by

Proposition 5. Let x^1 be an orthogonal separable coordinate for the massive HJ or KG (massless HJ or WE) and none of the coordinates (x^α) be adapted to a (conformal) Killing vector. Then

(i) a quadratic (conformal) Killing tensor exists;
(ii) the coordinate form of the separable coordinate is a closed eigenform of the (conformal) Killing tensor.

Proof. We give first the proof for the massive equations. The Hamiltonian H corresponding to (21) is given by

$$H = \frac{1}{2}\frac{H_1 + H_2}{U_1 + U_2}$$

with $H_1 := (p_1)^2$ and $H_2 := W^{\alpha\beta}(x^\gamma)\, p_\alpha p_\beta$ such that

$$K = \frac{1}{2}\frac{U_1 H_2 - U_2 H_1}{U_1 + U_2}$$

is a quadratic constant of motion because of proposition 2 and the Killing tensor

$$K^{ab} = (U_1 + U_2)^{-1}(U_1 W^{\alpha\beta}\, d^a_\alpha\, \delta^b_\beta - U_2\, \delta^a_1\, \delta^b_1)$$

generates K and fulfils, transvected with the closed form δ^1_a,

$$K^{ab}\,\delta^1_b = -U_2(U_1 + U_2)^{-1}\,\delta^a_1 \sim g^{ab}\,\delta^1_b = (U_1 + U_2)^{-1}\,\delta^a_1.$$

We find a similar proof for the massless equations. H_1 is a quadratic constant of motion for γ_{ab}. Therefore H_1 defines a conformal constant of motion for g_{ab} (see corollary 1) generated by the conformal Killing tensor $Q^{ab} = \delta^a_1\,\delta^b_1$ such that

$$Q^{ab}\,\delta^1_b = \delta^b_1 \sim g^{ab}\,\delta^1_b = U^{-1}\,\delta^a_1.$$

It is easy to realize that an orthogonal separable coordinate for the massless equations is also determined up to transformation (23). In particular, we find from the propositions 3 and 5.

Corollary 2. There are only orthogonal separable systems for the massive HJ and KG (massless HJ; WE) if (M, g) admits no isometry (conformal isometry; homothety).

4. Separable systems

We drop the assumption made in § 3 that none of the $n-1$ coordinates (x^α) is adapted to a (conformal) Killing vector, in order to find *all* separable systems for the equations considered. Let p coordinates (x^r) with $r, s = n-p+1, \ldots, n$ be adapted to p commuting Killing vectors (conformal Killing vectors; generators of homotheties) in the case of the massive HJ and KG (massless HJ; WE) where $p = 0, 1, 2, \ldots$. We know the unique *ansatz* for adapted coordinates and divide therefore the separation *ansatz* into:

(*a*) *Ansatz* (3) for the HJ takes the form

$$S(x^\alpha) = kx^1 + S_2(x^j) + k_r x^r \tag{24}$$

with arbitrary real constants k and k_r; *ansatz* (4) for the KG and WE takes the form

$$\Psi(x^\alpha) = \exp(cx^1)\chi(x^j)\exp(c_r x^r) \tag{25}$$

with arbitrary complex constants c and c_r. i, j, k run from 2 to $n-p$.

(*b*) *Ansatz* (3) for the HJ is

$$S(x^\alpha) = S_1(x^1) + S_2(x^j) + k_r x^r \qquad \text{with } S_1 \neq kx^1 \tag{26}$$

while *ansatz* (4) for the KG and WE is

$$\Psi(x^\alpha) = \Phi(x^1)\chi(x^j)\exp(c_r x^r) \qquad \text{with } \Phi \neq \exp(cx^1). \tag{27}$$

Case (a). We insert (24) into the HJ (5) and obtain

$$\frac{g^{ij}}{g^{11}}\partial_i S_2\,\partial_j S_2+2\left(\frac{g^{jr}}{g^{11}}k_r+\frac{g^{1j}}{g^{11}}k\right)\partial_j S_2+\frac{g^{rs}}{g^{11}}k_rk_s+2\frac{g^{1r}}{g^{11}}kk_r-\frac{m^2}{g^{11}}+k^2=0$$

where $\partial_r g^{ab}=0$ (or $\partial_r(g^{ab}/g^{11})=0$) because the coordinates (x^r) are adapted to commuting (conformal) Killing vectors. Therefore the massive (massless) HJ separates with respect to x^1 if and only if $\partial_1 g^{ab}=0$ (or $\partial_1(g^{ab}/g^{11})=0$) since k and k_r are arbitrary constants, and x^1 is adapted to a (conformal) Killing vector commuting with the given (conformal) Killing vectors. *Ansatz* (25) inserted into the KG (6) yields

$$\frac{1}{\chi}\left[\frac{g^{ij}}{g^{11}}\partial_i\,\partial_j+2\left(\frac{g^{jr}}{g^{11}}c_r+\frac{g^{1j}}{g^{11}}c\right)\partial_j+\frac{\partial_1(g^{1j}\sqrt{|g|})+\partial_i(g^{ij}\sqrt{|g|})}{g^{11}\sqrt{|g|}}\partial_j\right]\chi$$
$$+\frac{g^{rs}}{g^{11}}c_rc_s+2\frac{g^{1r}}{g^{11}}cc_r-\frac{m^2}{g^{11}}+c^2$$
$$+c\left(\partial_1(\ln|g^{11}\sqrt{|g|}|)+\frac{\partial_j(g^{1j}\sqrt{|g|})}{g^{11}\sqrt{|g|}}\right)+c_r\left(\frac{\partial_1(g^{1r}\sqrt{|g|})}{g^{11}\sqrt{|g|}}+\frac{\partial_j(g^{1r}\sqrt{|g|})}{g^{11}\sqrt{|g|}}\right)=0.$$

Because c and c_r are arbitrary constants, the KG separates with respect to x^1 if and only if $\partial_1 g^{ab}=0$, and the WE ($m=0$) if and only if

$$\partial_1(g^{ab}/g^{11})=0 \qquad \text{and} \qquad \partial_a\,\partial_1\ln|g^{11}|=0.$$

So we obtain for the equations considered one additional trivial separable coordinate if *ansatz* (24) or (25) is successful. (15) suggests that

$$\begin{aligned} x^1&=A^1(\bar{x}^1)+B^1(\bar{x}^\alpha)\\ x^j&=B^j(\bar{x}^\alpha)\\ x^r&=A^r(\bar{x}^1)+B^r(\bar{x}^\alpha)\end{aligned}\tag{28}$$

is the most general transformation such that $\bar{x}^1$ is again a separable coordinate for the transformed equations where A^1, A^r and B^r are arbitrary functions. $\bar{x}^1$ proves to be separable of type II. For $p=0$ the results agree with proposition 3. (28) implies

Proposition 6. All type II separable systems can be reduced to trivial separable systems.

Case (b). *Ansatz* (26) or (27) inserted into the corresponding equations gives rise to the terms $g^{1j}\,\partial_1 S_1\,\partial_j S_2$ or $g^{1j}\,\partial_1\Phi\,\partial_j\chi$ where S_1, Φ and S_2,χ are functions of x^1 or (x^j). Therefore $g^{1j}=0$ is a necessary condition for separability with respect to x^1. Then we find the HJ

$$(\partial_1 S_1)^2+2\frac{g^{1r}}{g^{11}}k_r\,\partial_1 S_1+\frac{g^{ij}}{g^{11}}\partial_i S_2\,\partial_j S_2+2k_r\frac{g^{jr}}{g^{11}}\partial_j S_2+k_rk_s\frac{g^{rs}}{g^{11}}-\frac{m^2}{g^{11}}=0\tag{29}$$

where $\partial_r g^{ab}=0$ (or $\partial_r(g^{ab}/g^{11})=0$). Because k and k_r are arbitrary constants and S_1, S_2 and their derivatives are independent functions, the massive HJ separates with respect to x^1 if

$$\begin{aligned} &g^{1j}=0\\ &\partial_j(g^{1r}/g^{11})=0; \qquad \partial_1(g^{ij}/g^{11})=0; \qquad \partial_1(g^{jr}/g^{11})=0\\ &\partial_1\,\partial_j(g^{rs}/g^{11})=0; \qquad \partial_1\,\partial_j(g^{11})^{-1}=0.\end{aligned}\tag{30}$$

For the massless HJ the last condition of (30) disappears because $m=0$. These conditions are sufficient, too.

For the KG we find with *ansatz* (27) and $g^{1j}=0$

$$\frac{1}{\Phi}\left(\partial_1\partial_1+2\frac{g^{1r}}{g^{11}}c_r\partial_1+\partial_1\ln|\sqrt{|g|}g^{11}|\partial_1\right)\Phi+\frac{1}{\chi}\left(\frac{g^{ij}}{g^{11}}\partial_i\partial_j+2\frac{g^{jr}}{g^{11}}c_r\partial_j+\frac{\partial_j(\sqrt{|g|}g^{ij})}{\sqrt{|g|}g^{11}}\partial_i\right)\chi$$

$$+\frac{g^{rs}}{g^{11}}c_rc_s+c_r\frac{\partial_1(\sqrt{|g|}g^{1r})+\partial_j(\sqrt{|g|}g^{jr})}{\sqrt{|g|}g^{11}}-\frac{m^2}{g^{11}}=0$$

which splits up in the required form if and only if the coefficient of Φ^{-1} is a function of x^1, the coefficient of χ^{-1} a function of (x^j) and the remaining term splits up into a sum of two terms depending on x^1 and (x^j), respectively. Therefore we find (30) and additionally

$$\partial_1\partial_j\ln|\sqrt{|g|}g^{11}|=0 \qquad (31)$$

to be necessary and sufficient conditions for separability of the KG with respect to x^1. For the WE the last condition of (30) drops ($m=0$). The transformation

$$x^1=x'^1; \qquad x^j=x'^j; \qquad x^r=x'^r+\int^{x'^1}(g^{1r}/g^{11})\,dx^1$$

changes the conditions (30), after dropping the dash, to

$$g^{1\alpha}=0 \qquad (32a)$$

$$\partial_1(g^{ij}/g^{11})=0 \qquad (32b)$$

$$\partial_1(g^{jr}/g^{11})=0 \qquad (32c)$$

$$\partial_1\partial_j(g^{rs}/g^{11})=0 \qquad (32d)$$

$$\partial_1\partial_j(g^{11})^{-1}=0. \qquad (32e)$$

(31) remains unchanged. We satisfy (32*b*, *c* and *d*) by

$$F^{ij}(x^k):=g^{ij}/g^{11}$$

$$F^{jr}(x^k):=g^{jr}/g^{11}$$

$$F_1^{rs}(x^1)+F_2^{rs}(x^k):=g^{rs}/g^{11}$$

where the F are arbitrary functions. Integration of (32*e*) yields

$$(g^{11})^{-1}=U_1(x^1)+U_2(x^k) \qquad (33)$$

with arbitrary functions U_1 and U_2. x^1 is an orthogonal coordinate because of (32*a*). Therefore we have

Theorem 2. The coordinates (x^r) are adapted to p commuting (conformal) Killing vectors. x^1 is an orthogonal separable coordinate for the massive (massless) HJ if and only if the metric takes the form

$$\left(\frac{\partial}{\partial s}\right)^2=(U_1(x^1)+U_2(x^k))^{-1}\left[\left(\frac{\partial}{\partial x^1}\right)^2+F^{ij}(x^k)\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\right.$$
$$\left.+2F^{jr}(x^k)\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^r}+(F_1^{rs}(x^1)+F_2^{rs}(x^k))\frac{\partial}{\partial x^r}\frac{\partial}{\partial x^s}\right] \tag{34}$$

(with the conformal factor $U=U(x^1, x^j)$).

For the KG (WE), (32) implies the same metrics but we have additionally to fulfil (31).

Theorem 3. Let p coordinates (x^r) be adapted to commuting Killing vectors (generators of homotheties). x^1 is an orthogonal separable coordinate for the KG (WE) if and only if the metric is conformal to

$$\gamma^{ab}\frac{\partial}{\partial x^a}\frac{\partial}{\partial x^b}=\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1}+F^{ij}(x^k)\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}$$
$$+2F^{jr}(x^k)\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^r}+(F_1^{rs}(x^1)+F_2^{rs}(x^k))\frac{\partial}{\partial x^r}\frac{\partial}{\partial x^s} \tag{35}$$

with the conformal factor $U_1(x^1)+U_2(x^j)$ (or $U(x_1, x_j)$), and $R_{1j}=0$ (or $R_{1j}=(4U)^{-1}(n-2)\,\partial_1\,\partial_j U$).

Proof. (34) is conformal to (35). The conformal factor is $V(x^1, x^j)$ such that $g^{ab}=V^{-1}\gamma^{ab}$. Let $\Omega^2:=|V|$. The Ricci tensors R_{ab} of g^{ab} and P_{ab} of γ^{ab} are related by

$$R_{ab}=P_{ab}+(n-2)\Omega\nabla_a\nabla_b\Omega^{-1}-(n-2)^{-1}\Omega^{-n+2}\nabla_c\nabla_d(\Omega^{n-2})\gamma^{cd}\gamma_{ab}.$$

With (32*a*) we find

$$R_{1j}=P_{1j}+(n-2)\Omega\nabla_1\nabla_j\Omega^{-1}.$$

Γ^a_{bc} and ∇_a are the Christoffel symbols and the covarant derivative of γ^{ab}. The Ricci tensor of γ^{ab} is given by

$$P_{1j}=-\tfrac{1}{2}\partial_1\,\partial_j\ln|\gamma|+\partial_a\Gamma^a_{1j}+\tfrac{1}{2}\Gamma^a_{1j}\partial_a\ln|\gamma|-\Gamma^a_{b1}\Gamma^b_{aj}$$

where $\gamma:=\det\gamma_{ab}$. The explicit form of (35) implies that the second and third terms of P_{1j} vanish, that $\Gamma^a_{1b}\Gamma^b_{ja}=1/4\,\partial_1\,\partial_j\ln|\gamma|$ and $\partial_1\,\partial_j\Omega=\nabla_1\nabla_j\Omega$. Therefore

$$R_{1j}=-\tfrac{3}{4}\partial_1\,\partial_j\ln|\gamma|+\tfrac{3}{4}(n-2)(\partial_1\ln|V|\,\partial_j\ln|V|-\tfrac{2}{3}V^{-1}\partial_1\,\partial_j V). \tag{36}$$

(31) is equivalent with $\partial_1\,\partial_j\ln|\Omega^{n-2}\sqrt{|\gamma|}|=0$ which leads to

$$\partial_1\,\partial_j\ln|\gamma|=(n-2)(V^{-1}\partial_1\,\partial_j V-\partial_1\ln|V|\,\partial_j\ln|V|).$$

The last equation together with (36) yields

$$R_{1j}=(n-2)(4V)^{-1}\,\partial_1\,\partial_j V.$$

$R_{1j}=0$ if (33) is valid. For $p=0$ we obtain the results of proposition 4.

Orthogonal separable systems and constants of motion are related by

Theorem 4. In (M, g) there exists an orthogonal separable system for the massive HJ and KG (massless HJ and WE). Then
(i) a quadratic (conformal) Killing tensor exists;
(ii) the coordinate form of the separable coordinate is an eigenform of the (conformal) Killing tensor.

Proof. Let x^1 be an orthogonal separable coordinate for the massive HJ or KG. The metric takes the form (34) (see theorems 2, 3) and its Hamiltonian is given by

$$H = \frac{1}{2}\frac{H_1 + H_2}{U_1 + U_2}$$

with $H_1 := (p_1)^2 + F_1^{rs}(x^1)p_r p_s$ and $H_2 := F^{ij}(x^k)p_i p_j + 2F^{jr}(x^k)p_j p_r + F_2^{rs}(x^k)p_r p_s$ such that (proposition 3)

$$K = \frac{1}{2}\frac{U_1 H_2 - U_2 H_1}{U_1 + U_2}$$

is a quadratic constant of motion generated by the Killing tensor

$$K^{ab} = \frac{U_1}{U}(F^{ij}\,\delta_i^a\,\delta_j^b + 2F^{jr}\,\delta_j^a\,\delta_r^b + F_2^{rs}\,\delta_r^a\,\delta_s^b) - \frac{U_2}{U}(\delta_1^a\,\delta_1^b + F_1^{rs}\,\delta_r^a\,\delta_s^b).$$

Contraction with the coordinate form δ_a^1 yields

$$K^{ab}\,\delta_b^1 = -\frac{U_2}{U}\,\delta_1^a \sim g^{ab}\,\delta_b^1 = U^{-1}\,\delta_1^a.$$

If x^1 is an orthogonal separable coordinate for the massless HJ or WE, H_1 is a quadratic constant of motion for (35). Therefore corollary 1 ensures that $Q^{ab} = \delta_1^a\,\delta_1^b + F_1^{rs}\,\delta_r^a\,\delta_s^b$ is a conformal Killing tensor fulfilling $Q^{ab}\,\delta_b^1 = \delta_1^a \sim g^{ab}\,\delta_b^1 = U^{-1}\,\delta_1^a$.

Finally, we construct all separable systems which can be derived from an orthogonal separable system. We give the most general transformation $x^a = x^a(\bar{x}^b)$ such that the transformed equations separate with respect to $\bar{x}^1$. The *ansatz* used is (26) or (27). Therefore the transformation is restricted by

$$\bar{\partial}_1\,\bar{\partial}_\alpha \bar{S}(\bar{x}^a) = 0 \qquad \text{or} \qquad \bar{\partial}_1\,\bar{\partial}_\alpha \ln \bar{\Psi}(\bar{x}^a) = 0$$

with $\bar{S}(\bar{x}^a) = S(x^b(\bar{x}^a))$, $\bar{\Psi}(\bar{x}^a) = \Psi(x^b(\bar{x}^a))$ and $\bar{\partial}_a = \partial/\partial\bar{x}^a$, which ensures that $\bar{S}$ or $\bar{\Psi}$ is of the form (3) or (4). That leads to

$$x^a = A^a(\bar{x}^1) + B^a(\bar{x}^\alpha) \qquad \text{with} \qquad \partial_1 A^1\,\partial_\alpha B^1 = 0 = \partial_1 A^i\,\partial_\alpha B^i.$$

If we investigate which combinations of vanishing factors are possible such that $\det(\partial x^a/\partial\bar{x}^b) \neq 0$, we find either: (i) $\partial_1 A^1 = 0$ and $\partial_1 A^j = 0$, so that

$$x^1 = B^1(\bar{x}^\alpha)$$
$$x^j = B^j(\bar{x}^\alpha)$$
$$x^r = A^r(\bar{x}^1) + B^r(\bar{x}^\alpha);$$

or (ii) $\partial_\alpha B^1 = 0$ and $\partial_1 A^j = 0$, so that

$$\begin{aligned} x^1 &= A^1(\bar{x}^1) \\ x^j &= B^j(\bar{x}^\alpha) \\ x^r &= A^r(\bar{x}^1) + B^r(\bar{x}^\alpha). \end{aligned} \tag{37}$$

The first transformation is included by (28) such that $\bar{x}^1$ is a separable coordinate of type II. Transformation (37) is a product of the two transformations:

$$\begin{aligned} x^1 &= A^1(\tilde{x}^1) \\ x^j &= \tilde{B}^j(\tilde{x}^j) \\ x^r &= A^r(\tilde{x}^1) + B^r(\tilde{x}^j) + C\tilde{x}^r \end{aligned} \tag{38}$$

with functions A^1, A^r and $\tilde{B}^j$, $\tilde{B}^r$, such that $\det(\partial x^a/\partial \tilde{x}^b) \neq 0$, and a constant $C \neq 0$, and

$$\begin{aligned} \tilde{x}^1 &= \bar{x}^1 \\ \tilde{x}^j &= \tilde{x}(\bar{x}^\alpha) \\ \tilde{x}^r &= \bar{x}^r \end{aligned} \tag{39}$$

where $(\tilde{x}^j)$ are arbitrary functions of $(\bar{x}^\alpha)$ and $\det(\partial\tilde{x}^i/\partial\bar{x}^j) \neq 0$. Then $B^j(\bar{x}^\alpha) := \tilde{B}^j(\tilde{x}^i(\bar{x}^\alpha))$ and $B^r(\bar{x}^\alpha) := \tilde{B}^r(\tilde{x}^j(\bar{x}^\alpha)) + C\bar{x}^r$. In particular, we find $\tilde{g}^{1j} = 0$ and after joining the second transformation (39)

$$\bar{g}^{1i}\frac{\partial\tilde{x}^j}{\partial\bar{x}^i} + \bar{g}^{1r}\frac{\partial\tilde{x}^j}{\partial\bar{x}^r} = 0$$

where $\bar{g}^{1r} = \tilde{g}^{1r} \sim \tilde{g}^{11} \sim g^{11} \neq 0$ and $\det(\partial\tilde{x}^i/\partial\bar{x}^j) \neq 0$. That is only to satisfy if

$$\frac{\partial\tilde{x}^j}{\partial\bar{x}^r} = 0 \leftrightarrow \tilde{x}^j = \tilde{x}^j(\bar{x}^i).$$

Therefore the product of (38) with (39) is reduced to a transformation of the form (38). A separable system related by (38) to an orthogonal separable system is called *separable of type I.*

Proposition 7. All separable systems of type I can be reduced to orthogonal separable systems.

The canonical metrics admitting an orthogonal separable coordinate are given in the theorems 2 and 3 where $p = 0, 1, 2, \ldots$. For $p = 0$ we find the results of proposition 5. (38) degenerates to (23).

5. Conclusions and final results

We have investigated all possibilities concerning the *ansatz* for the HJ, KG and WE and have found that two types of separable systems exist. Therefore we have extended Woodhouse's theorem 4.1 for the HJ to the KG and WE:

Theorem 5. All separable systems for the HJ, KG and WE in (M, g) are of type I or type II.

The type I separable systems are reducible to orthogonal separable systems (proposition 7) whose separable coordinate form is a closed eigenform of a (conformal) Killing tensor (theorem 4). The type II separable systems are reducible to trivial separable systems associated with local (conformal) isometry groups (theorem 1). We found canonical metrics admitting separable systems.

Finally, we derive the sufficient condition concerning constants of motion for the existence of separable systems. Proposition 3 shows that the existence of a Killing vector (conformal Killing vector; generator of a homothety) is sufficient for the existence of a trivial separable system for the massive HJ and KG (massless HJ; WE). Secondly, we consider orthogonal separable systems. For the HJ we quote theorem 4.2 of Woodhouse (1975).

Theorem 7. (M, g) admits $n-1$ quadratic (conformal) Killing tensors $\underset{\alpha}{K}^{ab}$ with the associated functions $\underset{\alpha}{K}$ such that:

(i) all $\underset{\alpha}{K}$ are linearly independent functions;

(ii) $\{\underset{\alpha}{K}, \underset{\beta}{K}\}=0$;

(iii) if all $\underset{\alpha}{K}^{ab}$ have the common closed eigenform say dx^1, then x^1 is an orthogonal separable coordinate for the massive (massless) HJ.

The proof is given in Woodhouse's paper. Theorem 2 ensures a choice of coordinates such that the metric agrees with (34). Because of theorem 3, x^1 is also an orthogonal separable coordinate for the KG and WE if additional conditions for the Ricci tensor are fulfilled.

Theorem 8. In (M, g) let p coordinates (x^r) with $r=n-p+1, \ldots, n$ be adapted to p commuting generators of local isometries (homotheties) and let $n-1$ quadratic (conformal) Killing tensors $\underset{\alpha}{K}^{ab}$ exist with the associated functions $\underset{\alpha}{K}$ such that:

(i) all $\underset{\alpha}{K}$ are linearly independent functions;

(ii) $\{\underset{\alpha}{K}, \underset{\beta}{K}\}=0$;

(iii) all $\underset{\alpha}{K}^{ab}$ have the common closed eigenform say dx^1;

(iv) $R_{1j}=0$ with $j=2, \ldots, n-p$ (or $R_{1j}=1/4g^{11}(n-2)\,\partial_1\,\partial_j(g^{11})^{-1}$), then x^1 is an orthogonal separable coordinate for the KG (WE).

We illustrate some results by a non-trivial example: The Kerr–Newman solution is, in Boyer–Lindquist coordinates $(x^a)=(\theta, r, \phi, t)$, given by

$$\left(\frac{\partial}{\partial s}\right)^2=(r^2+a^2\cos^2\theta)^{-1}\left[\left(\frac{\partial}{\partial\theta}\right)^2+\Delta\left(\frac{\partial}{\partial r}\right)^2+\left(\sin^{-2}\theta-\frac{a^2}{\Delta}\right)\left(\frac{\partial}{\partial\phi}\right)^2\right.$$
$$\left.+2a\left(1-\frac{r^2+a^2}{\Delta}\right)\frac{\partial}{\partial\phi}\frac{\partial}{\partial t}+\left(a^2\sin^2\theta-\frac{(r^2+a^2)^2}{\Delta}\right)\left(\frac{\partial}{\partial t}\right)^2\right] \tag{40}$$

where $\Delta=\Delta(r):=r^2-2mr+a^2+e^2$ and m, a, e are constants. ϕ and t are adapted to two commuting Killing vectors and are therefore trivial separable coordinates for the HJ, KG and WE. We compare with theorem 2: $x^1=\theta$, $x^2=r$ $(j=1, 2)$ and $(x^r)=(\phi, t)$ where $r=3, 4$. So (40) takes the canonical form (34) and θ is an orthogonal separable coordinate for the HJ. The Ricci tensor component $R_{\theta r}$ of (40) vanishes such that the KG and WE separate with respect to θ because of theorem 3. It is worth remarking that the HJ, KG and WE are also separable in the Kerr coordinates $(\tilde{x}^a) = (\tilde{\theta}, \tilde{r}, \tilde{\phi}, \tilde{t})$ because the transformation

$$
\begin{aligned}
\mathrm{d}\theta &= \mathrm{d}\tilde{\theta} \\
\mathrm{d}r &= \mathrm{d}\tilde{r} \\
\mathrm{d}\phi &= -a\Delta^{-1}\,\mathrm{d}\tilde{r} + \mathrm{d}\tilde{\phi} \\
\mathrm{d}t &= -(r^2+a^2)\Delta^{-1}\,\mathrm{d}\tilde{r} + \mathrm{d}\tilde{t}
\end{aligned}
$$

is exactly of the form (38) and $\tilde{\phi}$, $\tilde{t}$ are again adapted to the commuting Killing vectors.

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